Math 184 Exam 4 SHOW ALL WORK Name Key

1. Use the definition to show that $\mathcal{L}\{t\} = \frac{1}{s^2}$.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt \qquad u = t \quad v' = e^{-st}$$

$$u' = 1 \quad v = -\frac{1}{s}e^{-st}$$

$$= -\frac{1}{s} t e^{-st} \Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} \, dt$$

$$= -\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^\infty$$

$$= (0 - 0) - \left(0 - \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}$$

2. Find $L^{-1}\left\{\frac{s-3}{s^2+4}\right\}$. $= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2} \cdot \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$$= \cos 2t - \frac{3}{2}\sin 2t$$

3. Find $L\{e^{7t}t^5\}$. $= \frac{5!}{(s-7)^6} = \frac{120}{(s-7)^6}$

4. Use the Laplace transform to solve the IVP: $y' + 4y = e^t$, $y(0) = 7$

$$-7 + s\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$\mathcal{L}\{y\}(s+4) = 7 + \frac{1}{s-1}$$

$$\mathcal{L}\{y\} = \frac{7}{s+4} + \frac{1}{(s-1)(s+4)}$$

$$\frac{1}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4}$$

$$1 = A(s+4) + B(s-1)$$

$$1 = 5A \Rightarrow A = \frac{1}{5}$$

$$-1 = -5B \Rightarrow B = -\frac{1}{5}$$

$$\mathcal{L}\{y\} = \frac{1}{s+4} \cdot \frac{34}{5} + \frac{1}{s-1} \cdot \frac{1}{5}$$

$$y = \frac{34}{5} e^{-4t} + \frac{1}{5} e^t$$

5. Express using the Unit Step Function, then find $L\{f(t)\}$: $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ t-3, & t \ge 3 \end{cases}$

$$f(t) = (t-3)\,u_3(t)$$

$$\mathcal{L}\{f(t)\} = e^{-3s} \cdot \frac{1}{s^2}$$

6. Solve the IVP: $y' + 2y = \delta(t-3)$, $y(0) = 1$

$$-1 + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = e^{-3s}$$

$$\mathcal{L}\{y\}(s+2) = e^{-3s} + 1$$

$$\mathcal{L}\{y\} = \frac{1}{s+2}e^{-3s} + \frac{1}{s+2}$$

$$y = e^{-2(t-3)}u_3(t) + e^{-2t}$$

7. Solve for y(t): $y(t) = 3t^2 - e^{-t} - \int_0^t y(\tau)e^{t-\tau}\,d\tau$

$$\frac{s-1}{s-1} + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\mathcal{L}\{y\} = \frac{3\cdot 2}{s^3} - \frac{1}{s+1} - \mathcal{L}\{y\}\cdot\frac{1}{s-1}$$

$$\mathcal{L}\{y\}\left(1 + \frac{1}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$\mathcal{L}\{y\} = \frac{6}{s^3}\cdot\frac{s-1}{s} - \frac{1}{s+1}\left(\frac{s-1}{s}\right) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$y = 3t^2 - t^3 + 1 - 2e^{-t}$$

8. Solve the IVP using Laplace transforms: $y'' + 4y' + 13y = 0$, $y(0) = 0$, $y'(0) = 2$

$-2 + 0 \cdot s + s^2 \mathcal{L}\{y\} + 0 + 4s\mathcal{L}\{y\} + 13\mathcal{L}\{y\} = 0$

$\mathcal{L}\{y\}(s^2 + 4s + 13) = 2$

$\mathcal{L}\{y\} = \dfrac{2}{s^2+4s+4+9} = \dfrac{2}{(s+2)^2+3^2} \cdot \dfrac{3}{3}$

$y = \dfrac{2}{3} e^{-2t} \cdot \sin 3t$

9. Solve the system using Laplace transforms: $\begin{cases} x' = x + 3y \\ y' = 5x - y \end{cases}$, $x(0) = 2$, $y(0) = -1$

$-2 + s\mathcal{L}\{x\} = \mathcal{L}\{x\} + 3\mathcal{L}\{y\}$

$1 + s\mathcal{L}\{y\} = 5\mathcal{L}\{x\} - \mathcal{L}\{y\}$

$\mathcal{L}\{x\}(s-1) - 3\mathcal{L}\{y\} = 2$

$\mathcal{L}\{x\}(-5) + \mathcal{L}\{y\}(s+1) = -1$

$\mathcal{L}\{x\} \cdot 5(s-1) - 15\mathcal{L}\{y\} = 10$

$-\mathcal{L}\{x\}\, 5(s-1) + \mathcal{L}\{y\}(s-1)(s+1) = -1(s-1)$

$(s^2-16)\mathcal{L}\{y\} = 11 - s$

$\mathcal{L}\{y\} = \dfrac{11-s}{(s-4)(s+4)} = \dfrac{7}{8} \cdot \dfrac{1}{s-4} - \dfrac{15}{8} \cdot \dfrac{1}{s+4}$

$\boxed{y = \dfrac{7}{8} e^{4t} - \dfrac{15}{8} e^{-4t}}$

$11 - s = A(s+4) + B(s-4)$

$s = 4 \quad 7 = 8A \Rightarrow A = \dfrac{7}{8}$

$s = -4 \quad 15 = -8B \Rightarrow B = -\dfrac{15}{8}$

$5x = y + y'$

$x = \dfrac{1}{5}(y + y')$

$x = \dfrac{1}{5}\left(\dfrac{35}{8} e^{4t} + \dfrac{45}{8} e^{-4t}\right)$

$\boxed{x = \dfrac{7}{8} e^{4t} + \dfrac{9}{8} e^{-4t}}$

$y' = \dfrac{28}{8} e^{4t} + \dfrac{60}{8} e^{-4t}$