Math 184 Exam 4 SHOW ALL WORK Name Key

1. Use the definition to show that $L\{t\}=\frac{1}{s^2}$.

$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\cdot t\,dt$

$u=t \qquad v'=e^{-st}$

$u'=1 \qquad v=-\frac{1}{s}e^{-st}$

$= -\frac{1}{s}te^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$

$= \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty$

$= (0-0) - \left(0+\left(-\frac{1}{s^2}\right)\right)$

$= \frac{1}{s^2}$

2. Find $L^{-1}\left\{\frac{1}{s^2+64}\right\}$. $= \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8}{s^2+8^2}\right\} = \boxed{\frac{1}{8}\sin 8t}$

3. Find $L\{e^{-2t}t^4\}$. $= F(s+2)$ where $F(s) = \mathcal{L}\{t^4\} = \frac{4!}{s^5}$

$= \boxed{\frac{24}{(s+2)^5}}$

4. Use the Laplace transform to solve the IVP: $y' - 3y = e^{2t}$, $y(0) = 1$

$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$

$-y(0) + s\mathcal{L}\{y\} - 3\mathcal{L}\{y\} = \frac{1}{s-2}$

$(s-3)\mathcal{L}\{y\} = \frac{s-1}{s-2}$

$\mathcal{L}\{y\} = \frac{s-1}{(s-3)(s-2)} = \frac{2}{s-3} - \frac{1}{s-2}$

$s-1 = A(s-2) + B(s-3)$

$s=2$: $1 = -B \Rightarrow B = -1$

$s=3$: $2 = A$

$\boxed{y = 2e^{3t} - e^{2t}}$

5. Express using the Unit Step Function, then find $L\{f(t)\}$: $f(t) = \begin{cases} 3, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$

$\mathcal{L}\{3 - 3u_5(t)\} = \mathcal{L}\{3\} - 3\mathcal{L}\{u_5(t)\}$

$= \boxed{\frac{3}{s} - \frac{3e^{-5s}}{s}}$

6. Solve the IVP: $y' - 3y = \delta(t-2)$, $y(0) = 0$

$$-y(0) + s\mathcal{L}\{y\} - 3\mathcal{L}\{y\} = e^{-2s}$$

$$(s-3)\mathcal{L}\{y\} = e^{-2s}$$

$$\mathcal{L}\{y\} = e^{-2s}\cdot\frac{1}{s-3} \quad \& \quad \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

$$\boxed{y = u_2(t)\, e^{3(t-2)}}$$

7. Solve for y(t): $y(t) = 3t^2 - e^{-t} - \int_0^t y(\tau)e^{t-\tau}\,d\tau$

$$\mathcal{L}\{y\} = \frac{3\cdot 2}{s^3} - \frac{1}{s+1} - \mathcal{L}\{y\}\cdot\mathcal{L}\{e^t\}$$

$$\left(1 + \frac{1}{s-1}\right)\mathcal{L}\{y\} = 6\cdot\frac{1}{s^3} - \frac{1}{s+1}$$

$$s-1 = A(s+1) + B(s)$$

$$s=0:\ -1 = A$$

$$\frac{s}{s-1}\mathcal{L}\{y\} = 6\cdot\frac{1}{s^3} - \frac{1}{s+1}$$

$$s=-1:\ -2 = -B \Rightarrow B = 2$$

$$\mathcal{L}\{y\} = 6\cdot\frac{s-1}{s^4} - \frac{s-1}{s(s+1)} = 6\cdot\frac{1}{s^3} - 6\cdot\frac{1}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$= \frac{6}{2}\cdot\frac{2!}{s^3} - \frac{6}{6}\cdot\frac{3!}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$\boxed{y = 3t^2 - t^3 + 1 - 2e^{-t}}$$

8. Solve the IVP using Laplace transforms: $y'' + 5y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$

$-y'(0) - sy(0) + s^2\mathcal{L}\{y\} + 5(-y(0) + s\mathcal{L}\{y\}) + 4\mathcal{L}\{y\} = 0$

$-s - 5 + \mathcal{L}\{y\}(s^2 + 5s + 4) = 0$

$\mathcal{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+4)(s+1)}$

$s+5 = A(s+1) + (s+4)B$

$s=-1: \; 4 = 3B \Rightarrow B = \frac{4}{3}$

$s=-4: \; 1 = -3A \Rightarrow A = -\frac{1}{3}$

$\mathcal{L}\{y\} = -\frac{1}{3} \cdot \frac{1}{s+4} + \frac{3}{4} \cdot \frac{1}{s+1}$

$\boxed{y = -\frac{1}{3} e^{-4t} + \frac{4}{3} e^{-t}}$

9. Solve the system using Laplace transforms: $\begin{cases} x' = -x + y \\ y' = 2x \end{cases}$, $x(0) = 0$, $y(0) = 1$

$-0 + s\mathcal{L}\{x\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\}$

$-1 + s\mathcal{L}\{y\} = 2\mathcal{L}\{x\}$

$(s+1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 0$

$-2\mathcal{L}\{x\} + s\mathcal{L}\{y\} = 1$

$s(s+1)\mathcal{L}\{x\} - s\mathcal{L}\{y\} = 0$

$(s^2 + s - 2)\mathcal{L}\{x\} = 1$

$\mathcal{L}\{x\} = \frac{1}{(s+2)(s-1)} = -\frac{1}{3} \cdot \frac{1}{s+2} + \frac{1}{3} \cdot \frac{1}{s-1}$

$1 = A(s-1) + B(s+2)$

$s=1: \; 1 = 3B$

$s=-2: \; 1 = -3A$

$\boxed{x = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^{t}}$

$y' = 2x = -\frac{2}{3} e^{-2t} + \frac{2}{3} e^{t}$

$y = \frac{1}{3} e^{-2t} + \frac{2}{3} e^{t} + c$

$y(0) = 1 \Rightarrow 1 = \frac{1}{3} + \frac{2}{3} + c \Rightarrow c = 0$

$\boxed{y = \frac{1}{3} e^{-2t} + \frac{2}{3} e^{t}}$