Math 184 Exam 3 SHOW ALL WORK Name Key

1. Consider the transformation: $T: P_n \to P_{n+2}$ defined by $T((p(x)) = x^2 p(x)$.
   Is T a linear transformation? Prove that your answer is correct.

$$\text{(I)}\ T[p(x)+q(x)] = x^2[p(x)+q(x)] = x^2p(x)+x^2q(x) = T[p(x)]+T[q(x)]$$

$$\text{(II)}\ T[c\cdot p(x)] = x^2\cdot c\,p(x) = c\cdot x^2 p(x) = c\cdot T[p(x)]$$

So $\boxed{\text{yes}}$

2. Consider the linear transformation $T: R^3 \to R^3$ where $T(\vec{x}) = A\vec{x}$. Find the kernel of this transformation given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 9 & 12 \end{bmatrix}$$

Find $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = \vec{0}$!

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 9 & 12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \end{array}$$

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

3. Is the matrix $A = \begin{bmatrix} 5 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable? Explain.

$\lambda = 3$ (mult. 2), $\lambda = 5$

$$\lambda = 3: (\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & -2 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ x_1 = -x_2 \end{array} \qquad \vec{x} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

No. since only one e. vector for $\lambda = 3$ w/ mult. 2
(eigenvector)

4. For $\begin{bmatrix} 3 & 10 \\ 6 & -1 \end{bmatrix}$, find all eigenvalues and bases for the corresponding eigenspaces.

$\begin{vmatrix} \lambda-3 & -10 \\ -6 & \lambda+1 \end{vmatrix} = 0 \Rightarrow$ $\lambda^2 - 2\lambda - 63 = 0$ $(\lambda-9)(\lambda+7)=0$ $\lambda = 9, -7$

let $t=3$

$\lambda=9$ $\left[\begin{array}{cc|c} 6 & -10 & 0 \\ -6 & 10 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 6 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} \frac{5}{3}t \\ t \end{bmatrix} \to \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$

$\lambda=-7$ $\left[\begin{array}{cc|c} -10 & -10 & 0 \\ -6 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{x} \begin{bmatrix} -t \\ t \end{bmatrix} \to \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

5. Use your answer from problem 4 to solve the system $\vec{Y}' = A\vec{Y}$, where A is the matrix in problem 4.

$\vec{Y} = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} e^{9t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-7t}$

6. Suppose 3x3 matrix A has eigenvalues $\lambda = 5$ (multiplicity two) and $\lambda = -7$, with $\left\{ \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \right\}$ a basis for E_5, and with $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ a basis for E_{-7}. Find the matrix P that you would use to diagonalize A, and also find D such that $D = P^{-1}AP$.

$P = \begin{bmatrix} 1 & 1 & 2 \\ 6 & 6 & 2 \\ 7 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$\vec{Y}_1 \quad \vec{Y}_2$

7. Is it possible for $\vec{Y} = c_1\begin{bmatrix}12\\9\end{bmatrix}e^{8x} + c_2\begin{bmatrix}4\\3\end{bmatrix}e^{8x}$ to be the general solution to a second order system of first order HLDE's? Explain.

$\boxed{\text{NO}}$ since $\vec{Y}_1 = 3\vec{Y}_2$ so $\vec{Y}_1$ and $\vec{Y}_2$ are not L.I.

8. Solve: $\begin{aligned} y_1' &= 3y_1 - 18y_2 \\ y_2' &= 2y_1 - 9y_2 \end{aligned}$

$A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$ $\quad \begin{vmatrix} \lambda-3 & 18 \\ -2 & \lambda+9 \end{vmatrix} = 0 \Rightarrow$ $\lambda^2 + 6\lambda + 9 = 0$, $\lambda = -3$ (m-H.?)

$\left[\begin{array}{cc|c} -6 & 18 & 0 \\ -2 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{k} = \begin{bmatrix} 3t \\ t \end{bmatrix}$, use $\vec{k} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \leftarrow t = 1$

$(A - \lambda I)\vec{p} = \vec{k}$: $\left[\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \to \vec{p} = \begin{bmatrix} 3t + \frac{1}{2} \\ t \end{bmatrix}$ use $\vec{p} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \swarrow t = 0$

$$\boxed{\vec{Y} = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{-3t} + c_2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}te^{-3t} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}e^{-3t}\right)}$$

9. Consider the system: $\begin{aligned} y_1' &= 3y_1 - 18y_2 - 7x \\ y_2' &= 2y_1 - 9y_2 + 5e^x \end{aligned}$. Using your answer to problem 8 as a reference, find the Matrix M and vector $\vec{G}$ that you would use to solve the system.

$M = \begin{bmatrix} 3e^{-3t} & 3te^{-3t} + \frac{1}{2}e^{-3t} \\ e^{-3t} & te^{-3t} \end{bmatrix} \qquad \vec{G} = \begin{bmatrix} -7x \\ 5e^x \end{bmatrix}$

10. Consider the linear transformation $T: P_1 \to P_1$ where $T(ax+b) = ax + 4a - 2b$.
Find the eigenvalues and bases for the eigenspaces of this linear transformation, using $A = [T]_\alpha^\alpha$ where α is the standard basis for $P_1 : \alpha = \{x, 1\}$.

$T(x) = T(1 \cdot x + 0) = 1 \cdot x + 4 \cdot 1 - 2 \cdot 0 = 1x + 4 \cdot 1$
$T(1) = T(0 \cdot x + 1 \cdot 1) = 0 \cdot x + 4 \cdot 0 - 2 \cdot 1 = 0 \cdot x - 2 \cdot 1$ $\to A = \begin{bmatrix} 1 & 0 \\ 4 & -2 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & 0 \\ -4 & \lambda + 2 \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = -2, 1$

$\lambda = -2: \left[\begin{array}{cc|c} -3 & 0 & 0 \\ -4 & 0 & 0 \end{array}\right] \to \vec{P}_1 = \begin{bmatrix} 0 \\ t \end{bmatrix} \to$ use $t=1$ $\vec{P}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $0 \cdot x + 1 \cdot 1 = \boxed{1}$

$\lambda = 1: \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -4 & 3 & 0 \end{array}\right] \to \vec{P}_2 = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} \to$ use $t=4$ $\vec{P}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $3 \cdot x + 4 \cdot 1 = \boxed{3x+4}$

i.e. $T(1) = -2(1)$

$T(3x+4) = 1 \cdot (3x+4)$

Scores

9	9 6
8	5 4 6
7	4 4 0 3 8 7 6
6	5 6 3 7 0 3 2 4
5	7
4	7

11. Find one eigenvalue and one corresponding eigenvector of the linear transformation $T: R^2 \to R^2$ where T(x,y) consists of the reflection of the point (x,y) across the origin. Hint: you can use the definition $T(\vec{v}) = \lambda \vec{v}$ and just think graphically.

Any point reflective across origin will have the opposite signs

i.e. $T(x,y) = (-x,-y) = -1(x,y)$

So: $\lambda = -1$ with $\vec{v} = \langle x, y \rangle$