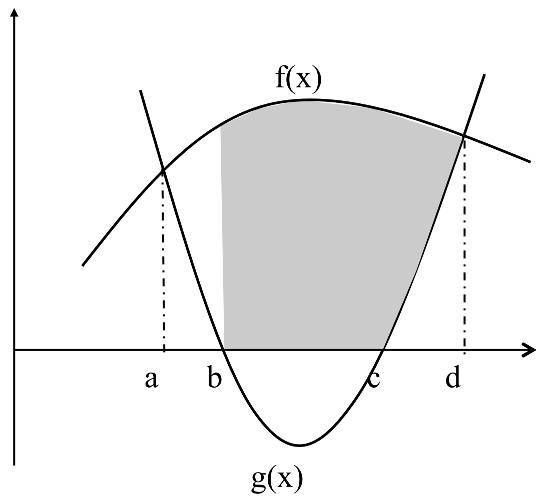
Math 182 Fall 2013 Final Exam Name________________________

1. Write the integral(s) that calculates the area of the **nonshaded (white)** region bounded by the curves.



2. Evaluate each definite or indefinite integral. Show all work in order to earn full credit.

a) $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$

b) $\int \frac{dx}{\sqrt{1-9x^2}}$

c) $\int x \ln x \, dx$

d) $\int \frac{5}{x^2 - x - 6}\, dx$

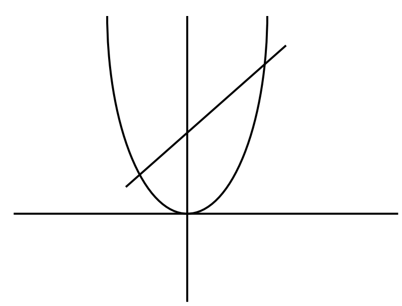
3. Determine whether each integral is convergent or divergent. If the integral converges, tell what it converges to. Show all work to earn full credit.

a) $\int_0^{\infty} 4te^{-t^2}\, dt$

b) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

4. Consider the region bounded by the curves $y = x^2$ and $y = x + 2$

SET UP the integral(s) to find:



(Note: Picture is not to scale.)

a) The volume of the solid formed

by rotating the region about the line $x = 5$.

DO NOT EVALUATE THE INTEGRAL.

b) The arc length of the boundary of the enclosed region.

DO NOT EVALUATE THE INTEGRAL.

5a) For the point with Cartesian coordinates $(\sqrt{3},1)$ write the point in its equivalent polar form.

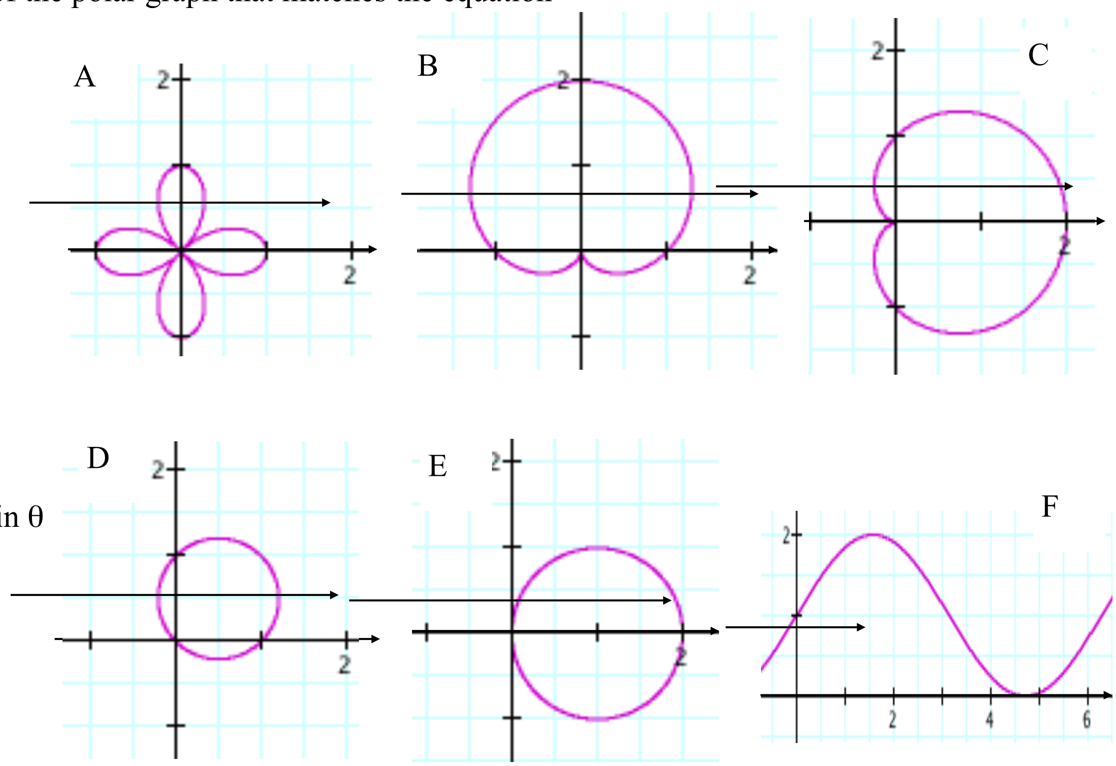
5b) For the point with polar form $(\sqrt{2}, \pi/4)$, give its equivalent Cartesian coordinate.

5c) Choose the letter of the polar graph that matches the equation

_______ $r = 1 + \sin\theta$

_______ $r = \cos 2\theta$

_______ $r = \cos\theta + \sin\theta$



6. Find the solution to the initial value problem: $x\dfrac{dy}{dx} = y^2, \qquad y(1) = -1/2$

7. The onset of an influenza epidemic is modeled by the equation $dP/dt = kP$ where $P(t)$ is the number of infected people at time t (in days). Suppose the epidemic begins with one case on Day 0 and that there are 20 cases one week later on Day 7.

a) Find the growth rate constant k, accurate to three decimal places.

b) When will the epidemic reach 100 cases? Give your answer correct to the nearest day.

8. Tell if each series converges or diverges, and which test you used to determine this. (There may be more than one correct letter.) Only write in column 3 if your answer is letter B, C or F.

	Converge or diverge?	Letter of Test used	function or ratio
a. $\sum_{n=1}^{\infty} \cos(\frac{\pi}{n})$	________	________	________
b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+\pi}}$	________	________	________
c. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$	________	______	________
d. $\sum_{n=0}^{\infty} 5\frac{(-4)^n}{(7)^{n+1}}$	________	________	________

A. nth term test

B. Comparison to (name function)

C. Limit comparison to (name function)

D. Ratio test

E. Alternating series test

F. Geometric series (name r)

9. Find all values of x for which this series converges: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)4^n}$

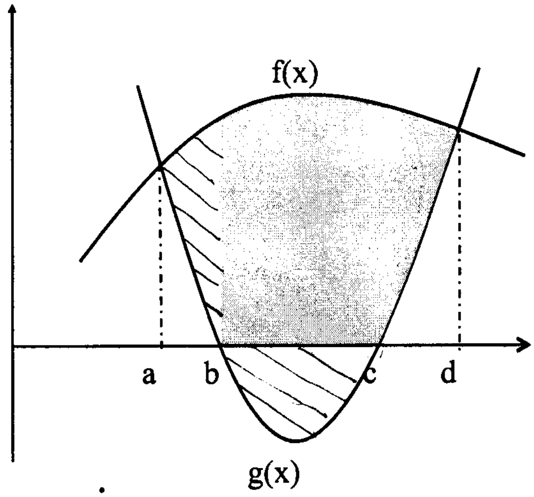
10.a) Write the Maclaurin polynomial of order 3 (i.e., the first four terms) for $f(x) = \frac{1}{x+1}$

b) Write the first 4 terms of the Maclaurin series for $f(x) = \ln(x+1)$
(Hint: you may be able to use your work from part a) .)

Math 182 Fall 2013 Final Exam Name________________________

1. $\boxed{\text{Confirmed}}$

Write the integral(s) that calculates the area of the **nonshaded (white)** region bounded by the curves.



$$\int_a^d [f(x)-g(x)]\,dx - \int_b^c f(x)\,dx - \int_c^d [f(x)-g(x)]\,dx$$

2. Evaluate each definite or indefinite integral. Show all work in order to earn full credit.

$\boxed{\text{Confirmed}}$

u-sub.

a) $\displaystyle\int_0^{\sqrt{\pi}} x \sin\left(x^2\right) dx$

$u = x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} 2x \sin(x^2)\,dx$$

$$= \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \sin(u)\,du$$

$$= \frac{1}{2}\left[-\cos(u)\right]_{x=0}^{x=\sqrt{\pi}}$$

$$= \frac{1}{2}\left[-\cos(x^2)\right]_0^{\sqrt{\pi}}$$

$$= \frac{1}{2}\left[(-\cos((\sqrt{\pi})^2)) - (-\cos(0))\right]$$

$$\to \frac{1}{2}\left[(-\cos(\pi)) - (-\cos(0))\right]$$

$$= \frac{1}{2}\left[1-(-1)\right]$$

$$= \frac{1}{2}\left[2\right]$$

$$= \boxed{1}$$

Trig Sub.

b) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

See attached.

$\boxed{\text{Confirmed}}$

Int. by parts

c) $\displaystyle\int x \ln x\,dx$ $\boxed{\text{Confirmed}}$

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^2}{2}$

$\int u\,dv = uv - \int v\,du$

$$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\left[\frac{1}{2}x^2\right] + C$$

$$\to \boxed{\frac{x^2 \ln x}{2} - \frac{1}{4}x^2 + C}$$

$\boxed{\text{Confirmed}}$

Partial Fractions

d) $\displaystyle\int \frac{5}{x^2-x-6}\,dx = \int \frac{5}{(x-3)(x+2)}\,dx$

$$\frac{5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$5 = A(x+2) + B(x-3)$$

Let $x=-2$: $5 = B(-5) \Rightarrow B = -1$

Let $x=3$: $5 = 5A \Rightarrow A = 1$

$$\int \frac{5}{(x-3)(x+2)}\,dx = \int \left[\frac{1}{x-3} + \frac{-1}{x+2}\right] dx$$

$$= \boxed{\ln|x-3| - \ln|x+2| + C}$$

Confirmed

3. Determine whether each integral is convergent or divergent. If the integral converges, tell what it converges to. Show all work to earn full credit.

a) $\int_0^\infty 4te^{-t^2}\,dt$

$u = -t^2$
$du = -2t\,dt$
$-\frac{1}{2}du = t\,dt$

$= \lim_{b\to\infty} \int_0^b 4te^{-t^2}\,dt = \lim_{b\to\infty} \int_{t=0}^{t=b} 4(-\tfrac{1}{2})e^u\,du$

$-2\left(\lim_{b\to\infty} \left[e^u\right]_{t=0}^{t=b}\right) = -2\left(\lim_{b\to\infty}\left[e^{-t^2}\right]_0^b\right) = -2\lim_{b\to\infty}\left(e^{-b^2} - e^0\right)$

$= -2\lim_{b\to\infty}\left(\frac{1}{e^{b^2}} - 1\right) = -2(0-1) = \boxed{2}$ Converges

Confirmed

b) $\int_0^1 \frac{dx}{\sqrt{1-x}}$ Discontinuous at $x=1$

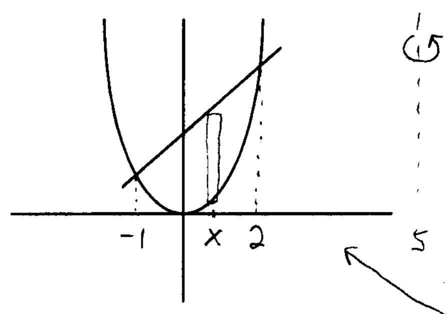
$= \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}}$ $\quad \left\{ \begin{array}{l} u = 1-x \\ du = -1\,dx \\ -du = dx \end{array} \right\}$

$= \lim_{t\to 1^-} \int_{x=0}^{x=t} \frac{-du}{\sqrt{u}} = \lim_{t\to 1^-}\left[-\int_{x=0}^{x=t} u^{-1/2}\,du\right] = \lim_{t\to 1^-}\left[-\frac{u^{1/2}}{1/2}\right]_{x=0}^{x=t}$

$= \lim_{t\to 1^-}\left[-2\sqrt{1-x}\right]_0^t = \lim_{t\to 1^-}\left[-2\sqrt{1-t} + 2\right] = -2\sqrt{1-1} + 2 = \boxed{2}$

4. Consider the region bounded by the curves $y = x^2$ and $y = x + 2$

SET UP the integral(s) to find: Confirmed



a) The volume of the solid formed by rotating the region about the line $x = 5$.

DO NOT EVALUATE THE INTEGRAL.

(Note: Picture is not to scale.)

radius $= 5-x$, circum. $= 2\pi(5-x)$, height $= (x+2) - x^2$

$$\boxed{\int_{-1}^{2} 2\pi(5-x)\left[(x+2)-x^2\right]dx}$$

Find points of intersection:

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

At $x=-1$ and $x=2$.

b) The arc length of the boundary of the enclosed region.

DO NOT EVALUATE THE INTEGRAL.

$$\boxed{\int_{-1}^{2}\sqrt{1+[2x]^2}\,dx + \int_{-1}^{2}\sqrt{1+[1]^2}\,dx}$$

where $\frac{d}{dx}\left[x^2\right] = 2x$ and $\frac{d}{dx}\left[x+2\right] = 1$.

$r^2 = x^2 + y^2$

$r^2 = (\sqrt{3})^2 + 1^2$

$r^2 = 4$

$r = 2$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

confirmed

$(r, \theta) = \boxed{(2, \frac{\pi}{6})}$

5a) For the point with Cartesian coordinates ($\sqrt{3}$,1) write the point in its equivalent polar form.

$x = r\cos\theta$

$x = \sqrt{2}\cos\frac{\pi}{4}$

$x = \sqrt{2}\cdot\frac{\sqrt{2}}{2}$

$x = 1$

$y = r\sin\theta$

$y = \sqrt{2}\sin\frac{\pi}{4}$

$y = \sqrt{2}\cdot\frac{\sqrt{2}}{2}$

$y = 1$

confirmed

$(x, y) = \boxed{(1, 1)}$

5b) For the point with polar form ($\sqrt{2}$, $\pi/4$), give its equivalent Cartesian coordinate.

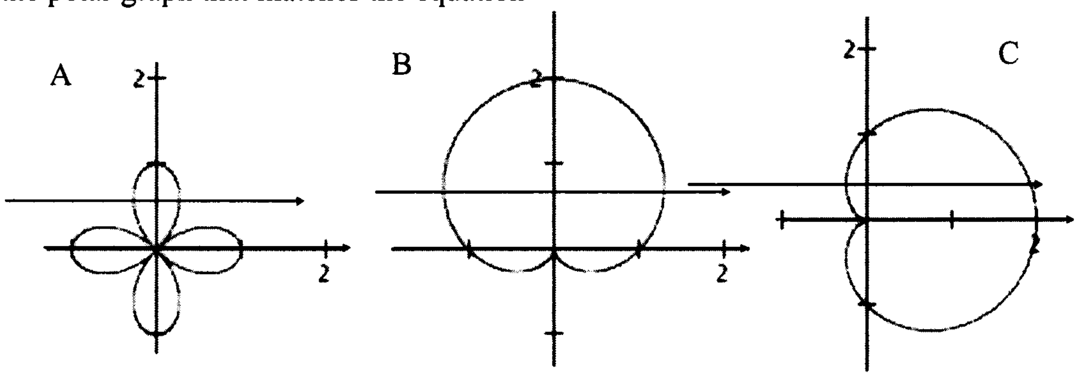
Confirmed

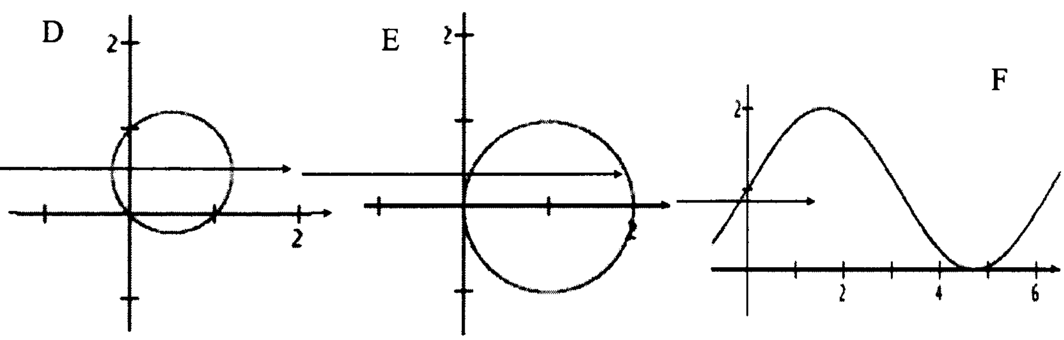
5c) Choose the letter of the polar graph that matches the equation

B $r = 1 + \sin\theta$

A $r = \cos 2\theta$

D $r = \cos\theta + \sin\theta$





Confirmed

6. Find the solution to the initial value problem: $x\frac{dy}{dx} = y^2$, $y(1) = -1/2$

$\frac{1}{y^2}\,dy = \frac{1}{x}\,dx$

$\int y^{-2}\,dy = \int \frac{1}{x}\,dx$

$\frac{y^{-1}}{-1} = \ln|x| + c$

$-\frac{1}{y} = \ln|x| + c$

Substitute $x = 1$, $y = -\frac{1}{2}$:

$\frac{-1}{-1/2} = \ln|1| + c$

$2 = 0 + c$

$c = 2$

So we have:

$\frac{-1}{y} = \ln|x| + 2$

$\boxed{y = \frac{-1}{\ln|x| + 2}}$

7. The onset of an influenza epidemic is modeled by the equation $dP/dt = kP$ where $P(t)$ is the number of infected people at time t (in days). Suppose the epidemic begins with one case on Day 0 and that there are 20 cases one week later on Day 7.

Confirmed

a) Find the growth rate constant k, accurate to three decimal places.

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P}dP = k\,dt$$

$$\int \frac{1}{P}dP = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C}$$

$$|P| = e^{kt}e^{C}$$

$$P = Ae^{kt}$$

$$P(t) = P_0 e^{kt}$$

using $P(t) = P_0 e^{kt}$:

$$P(t) = 1e^{kt}$$

$$20 = e^{k(7)}$$

$$\ln 20 = 7k$$

$$k = \frac{\ln 20}{7} \approx \boxed{0.428}$$

b) When will the epidemic reach 100 cases? Give your answer correct to the nearest day.

Confirmed

$$P(t) = P_0 e^{kt}$$

$$P(t) = 1e^{0.428t}$$

So we have

$$100 = e^{0.428t}$$

$$\ln 100 = 0.428t$$

$$t = \frac{\ln 100}{0.428} \approx 10.8$$

$$\boxed{11\text{th day}}$$

8. Tell if each series converges or diverges, and which test you used to determine this. (There may be more than one correct letter.) Only write in column 3 if your answer is letter B, C or F.

	Converge or diverge?	Letter of Test used	function or ratio
confirmed a. $\sum_{n=1}^{\infty} \cos(\frac{\pi}{n})$	Diverges	A	
confirmed b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+\pi}}$	Diverges	C	$\frac{1}{\sqrt{n}}$ (Divergent p-series)
confirmed c. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$	Converges	B	
confirmed d. $\sum_{n=0}^{\infty} 5\frac{(-4)^n}{(7)^{n+1}}$	Converge	F	$r = -\frac{4}{7}$

$= \sum_{n=0}^{\infty} \left(\frac{5}{7}\right)\left(\frac{-4}{7}\right)^n$

A. nth term test (Divergence Test)

B. Comparison to (name function)

C. Limit comparison to (name function)

D. Ratio test

E. Alternating series test

F. Geometric series (name r)

confirmed 9. Find all values of x for which this series converges: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)4^n}$

Using Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{[(n+1)+1]4^{n+1}} \cdot \frac{(n+1)4^n}{(x-3)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(x-3)}{4} \cdot \frac{n+1}{n+2}\right| = \frac{|x-3|}{4}.$

Converges when $\frac{|x-3|}{4} < 1 \Rightarrow |x-3| < 4 \Rightarrow -4 < x-3 < 4 \Rightarrow$

$-1 < x < 7.$

Check endpoint $x=-1$:

$\sum \frac{(-1-3)^n}{(n+1)4^n} = \sum \frac{(-4)^n}{(n+1)4^n} = \sum \frac{(-1)^n 4^n}{(n+1)\ 4^n} = \sum \frac{(-1)^n}{n+1}$

Alternating Series Test where $b_n = \frac{1}{n+1} > 0$

① $\frac{1}{(n+1)+1} \le \frac{1}{n+1}$ ✓ ② $\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓ Converges

Check endpoint $x=7$:

$\sum \frac{(7-3)^n}{(n+1)4^n} = \sum \frac{4^n}{(n+1)4^n} = \sum \frac{1}{n+1}$

Limit Comparison Test

Compare to divergent, harmonic serie $\sum \frac{1}{n}$.

$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$ an a finite num

diverges

Answer: $-1 \le x < 7$ or $[-1, 7)$

(Assuming we can't use the Binomial Series)

Confirmed

10.a) Write the Maclaurin polynomial of order 3 (i.e., the first four terms) for $f(x) = \dfrac{1}{x+1}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(x+1)^{-1}$	1
1	$-(x+1)^{-2}$	-1
2	$2(x+1)^{-3}$	2
3	$-6(x+1)^{-4}$	-6

$$f(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{1}{0!}x^0 + \frac{-1}{1!}x^1 + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3$$

$$= \boxed{1 - x + x^2 - x^3}$$

Confirmed

b) Write the first 4 terms of the Maclaurin series for f(x) = ln (x + 1)
(Hint: you may be able to use your work from part a) .)

From part (a) we have:

$$\frac{1}{x+1} = 1 - x + x^2 - x^3$$

Since $f(x) = \ln(x+1) = \int \frac{1}{x+1}\,dx$, we have

$$f(x) = \ln(x+1) = \int (1 - x + x^2 - x^3)\,dx$$

$$= \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$$

Problem # 2b $\quad \displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$ (Trig Substitution)

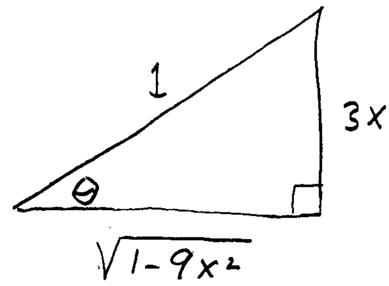
Note: $\sqrt{1-9x^2}$ somewhat resembles $\sqrt{a^2-x^2}$ which instructs use to use $x = a\sin\theta$ and $1-\sin^2\theta = \cos^2\theta$.

$$\left.\begin{aligned} \text{let } x &= \tfrac{1}{3}\sin\theta \\ dx &= \tfrac{1}{3}\cos\theta\, d\theta \end{aligned}\right\}$$

Since $x = \frac{1}{3}\sin\theta$

then $\sin\theta = 3x$

and $\theta = \sin^{-1}(3x)$



$$\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{\frac{1}{3}\cos\theta\, d\theta}{\sqrt{1-9\left(\frac{1}{3}\sin\theta\right)^2}} = \frac{1}{3}\int \frac{\cos\theta}{\sqrt{1-9\left(\frac{1}{9}\sin^2\theta\right)}}\, d\theta$$

$$= \frac{1}{3}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta = \frac{1}{3}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta = \frac{1}{3}\int 1\, d\theta$$

$$= \frac{1}{3}\theta + C = \boxed{\frac{1}{3}\sin^{-1}(3x) + C}$$